Fourier Series

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Overview

Fourier series are series of cosine and sine terms and arise in the important practical task of representing general periodic functions. They constitute a very important tool in solving problems that involve ordinary and pratical differential equations.

The theory of Fourier series is rather complicated, but the application of these series is simple. Fourier series are, in a certain sense, more universal than Taylor series, because many discontinuous periodic functions of practical interest can be developed in Fourier series, but, of course, do not have Taylor series representations.

Jean-Baptiste Joseph Fourier (1768 – 1830)

Jean-Baptiste Joseph Fourier, French physicist and mathematician, lived and taught in Paris, accompanied Napoleon to Egypt, and was later made prefect of Grenoble.

He utilized Fourier series in his main work *Theorie analytique de la chaleur (Analytic Theory of Heat, Paris 1822)*, in which he developed the theory of heat conduction (heat equation).

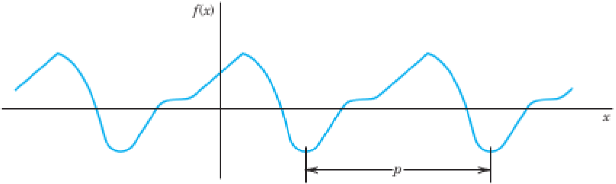
These new series became a most important tool in mathematical physics and also had considerable influence on the further development of mathematics itself.

Periodic Functions

A function $f(x)$ is called **periodic** if it is defined for all real x and if there is some positive number p such that

$$f(x+p) = f(x) \qquad \textit{for all } x. \tag{1}$$

This number p is called a **period** of $f(x)$. The graph of such a function is obtained by periodic repetition of its graph any interval of length p.



Periodic function of period p

Periodic Functions

Familiar periodic functions are the sine and cosine functions. We note that the function $f = c = const$ is also a periodic function in the sense of the definition, because it satisfies (1) for every positive p.

Examples of functions that are not periodic are $x, x^2, x^3, e^x, \cosh x$, and $\ln x$, to mention just a few.

From (1) we have $f(x + 2p) = f[(x + p) + p] = f(x + p) = f(x)$, etc., and for any integer n,

$$f(x + np) = f(x) \qquad \textit{for all } x.$$

Periodic Functions

Hence $2p, 3p, 4p, \dots$ are also periods of $f(x)$. Furthermore, if $f(x)$ and $g(x)$ have period p, then the function

$$h(x) = af(x) + bg(x), \text{for any constants } a, b.$$

also has the period p.

If a periodic function $f(x)$ has a smallest period $p\,(> 0)$, this is often called the **fundamental period** of $f(x)$.

For $\cos x$ and $\sin x$ the fundamental period is 2π, for $\cos 2x$ and $\sin 2x$ it is π, and so on.

A function without fundamental period is $f = const$.

Trigonometric Series

Our problem is to find a representation of various functions of period $p = 2\pi$ in terms of the simple functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \cdots. \tag{2}$$

These functions have the period 2π. The series that will arise in this connection will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots, \tag{3}$$

where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants.

Such a series is called a **trigonometric series** and a_n and b_n are called the **coefficients** of the series.

Trigonometric Series

Using the summation sign, we may write this series

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right). \tag{4}$$

The set of functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \cdots$ from which we have made up the series (4) if often called the **trigonometric system**, to have a short name for it. We see that each term of the series (4) has the period 2π. Hence if the series (4) converges, its sum will be a function of period 2π.

The point is that trignometric series can be used for representing any practically important periodic function f, simple or complicated, of any period p. (This series will then be called the **Fourier series** of f.)

Exercises

1. **Fundamental Period**. Find the smallest positive period p of the following functions :
   $\cos x, \sin x, \cos 2x, \sin 2x, \cos \pi x, \sin \pi x, \cos 2\pi x, \sin 2\pi x,$
   $\cos nx, \sin nx, \cos \frac{2\pi x}{k}, \sin \frac{2\pi x}{k}, \cos \frac{2\pi nx}{k}, \sin \frac{2\pi nx}{k}.$
2. If $f(x)$ and $g(x)$ have period p, show that $h = af + bg$ (a, b constant) has the period p.
3. **(Integer multiples of period)** If p is a period of $f(x)$, show that $np, n = 2, 3, \cdots$, is a period of $f(x)$.
4. **(Constant)** Show that the function $f(x) = const$ is a periodic function of period p for every positive p.
5. **(Change of Scale)** If $f(x)$ is a periodic function x of period p. Show that $f(ax), a \neq 0$, is a periodic function of x of period p/a, and $f(x/b) \neq 0$, is a periodic function of x of period bp. Verify these results for $f(x) = \cos x, a = b = 2$.

Sketch or plot the following functions $f(x)$, which are summed to be periodic with period 2π and, for $-\pi < x < \pi$, are given by the formulas

1. $f(x) = x$
2. $f(x) = x^2$
3. $f(x) = |x|$
4. $f(x) = \pi - |x|$
5. $f(x) = |\sin x|$
6. $f(x) = e^{-|x|}$

Sketch or plot the following functions $f(x)$, which are summed to be periodic with period 2π and, for $-\pi < x < \pi$, are given by the formulas

1. $f(x) = \begin{cases} x & \text{if } -\pi \le x \le 0 \\ 0 & \text{if } 0 \le x \le \pi \end{cases}$
2. $f(x) = \begin{cases} 0 & \text{if } -\pi \le x \le 0 \\ x^2 & \text{if } 0 \le x \le \pi \end{cases}$
3. $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$
4. $f(x) = \begin{cases} x & \text{if } 0 < x < \pi \\ \pi - x & \text{if } -\pi < x < 0 \end{cases}$
5. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ e^{-x} & \text{if } 0 < x < \pi \end{cases}$
6. $f(x) = \begin{cases} x^2 & \text{if } -\pi < x < 0 \\ -x^2 & \text{if } 0 < x < \pi \end{cases}$

Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions. That is,

$$a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \text{ for } N = 1, 2, 3, \ldots.$$

These series are trigonometric series whose coefficient are determined from $f(x)$ by the "Euler formulas," which we shall derive first.

Afterwards we shall take a look at the theory of Fourier series.

Euler Formulas for the Fourier Coefficients

Let us assume that $f(x)$ is a periodic function of period 2π and is integrable over a period. Let us further assume that $f(x)$ can be **represented** by a trigonometric series.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx); \tag{5}$$

that is, we assume that this series converges and has $f(x)$ as its sum.

Given such a function $f(x)$, we want to determine the coefficient a_n and b_n of the corresponding series (5).

Determination of the constant term a_0

Integrating on both sides of (5) from $-\pi$ to π we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx.$$

If term-by-term integration of the series is allowed, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \; dx + b_n \int_{-\pi}^{\pi} \sin nx \; dx \right).$$

The first term on the right equals $2\pi a_0$. All the other integrals on the right are zero, as can be readily seen by integration. Hence our first result is

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx. \qquad (6)$$

Determination of the coefficients a_n of the cosine terms.

Similarly, we multiply (5) by $\cos mx$, where m is any fixed positive integer, and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \cos mx \; dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx \; dx. \tag{7}$$

Integrating term by term, we see that the right side becomes

$$a_0 \int_{-\pi}^{\pi} \cos mx \; dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \;\; \cos mx \; dx + b_n \int_{-\pi}^{pi} \sin nx \;\; \cos mx \; dx \right]$$

The first integral is zero.

Determination of the coefficients a_n of the cosine terms.

Also

$$\int_{-\pi}^{\pi} \cos nx \; \cos mx \; dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x \; dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x \; dx,$$

$$\int_{-\pi}^{\pi} \sin nx \; \cos mx \; dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x \; dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x \; dx.$$

Integration shows that the four terms on the right are zero, except for the last term in the first line, which equals π when $n = m$. Since (7) this term is multiplied by a_m, the right side in (7) equals $a_m\pi$. Our second result is

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx \; dx, \qquad m = 1, 2, \cdots \tag{8}$$

Determination of the coefficients b_n of the sine terms.

We finally multiply (1) by $\sin mx$, where m is any fixed integer, and then integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \sin mx \ dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \sin mx \ dx. \tag{9}$$

Integrating term by term, we see that the right side becomes

$$a_0 \int_{-\pi}^{\pi} \sin mx \ dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \ \sin mx \ dx \ + b_n \int_{-\pi}^{pi} \sin nx \ \sin mx \ dx \right].$$

]

The first integral is zero. The next integral is of the kind considered before, and is zero for all $n = 1, 2, \cdots$. For the last integral we obtain

$$\int_{-\pi}^{\pi} \sin nx \ \sin mx \ dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos{(n-m)\, x} \ dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos{(n+m)\, x} \ dx.$$

The last term is zero. The first term on the right is zero when $n \neq m$ and is π when $n = m$.

Since in (9) this term is multiplied by b_m, the right side in (9) is equal to $b_m\pi$ and our last result is

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \ dx, \quad m = 1, 2, \cdots .$$

Summary of These Calculations : Fourier Coefficients, Fourier Series

From the derivations above, we have the so-called **Euler formulas**

$$(a) \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$(b) \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \ dx \qquad n = 1, 2, \ldots$$

$$(c) \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ dx \qquad n = 1, 2, \ldots$$

These numbers given by Euler formulas are called the **Fourier coefficients** of $f(x)$.

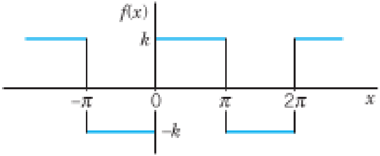
The trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{10}$$

with coefficients given above is called the **Fourier series** of $f(x)$.

Example 1 (Rectangular wave).

Find the Fourier coefficients of the periodic function $f(x)$ in the following figure.



Given function $f(x)$ (Periodic reactangular wave)

The formula is $f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$ *and* $f(x + 2\pi) = f(x)$. *Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)*

Rectangular wave

Solution. Since the area under the curve of $f(x)$ between $-\pi$ and π is zero, so $a_0 = 0$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \ dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}(-k)\cos nx \ dx + \int_0^{\pi} k\cos nx \ dx\right]$$
$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\bigg|_{\pi}^{0} + k\frac{\sin nx}{n}\bigg|_{0}^{\pi}\right] = 0$$

Because $\sin nx = 0$ at $-\pi, 0,$ and π for all $n = 1, 2, \cdots$. Similarly, we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx \ dx + \int_0^{\pi} k \ \sin nx \ dx\right]$$
$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - k\frac{\cos nx}{n}\bigg|_{\pi}^{0}\right]$$

Rectangular wave

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1 - \cos n\pi)$$

Now, $\cos\pi = -1, \cos 2\pi = 1, 3\pi = -1$, etc; in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd n,} \\ 1 & \text{for even n,} \end{cases} \text{ and thus } 1 - \cos n\pi = \begin{cases} 2 & \text{for odd n.} \\ 0 & \text{for even n.} \end{cases}$$

Hence the Fourier coefficients b_n of our function are

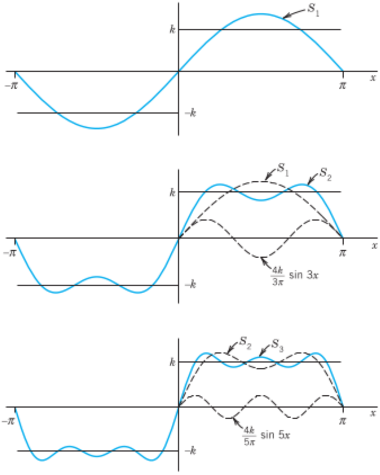
$b_1 = \frac{4k}{\pi}, b_2 = 0, b_3 = \frac{4k}{3\pi}, b_4 = 0, b_5 = \frac{4k}{5\pi}, \cdots$

Since the a_n are zero, the series of $f(x)$ is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right). \tag{11}$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \qquad etc.$$



First three partial sums of the corresponding Fourier series

Rectangular wave

Their graphs in the above figure seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the values $-k$ and k of our function. Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right).$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

This is a famous result by Leibniz (obtained in 1673 form geometrical considerations.) It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points.

Orthogonality of the Trigonometric System

The **trigonometric system**

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \cdots$$

is **orthogonal** on the interval $-\pi \leq x \leq \pi$ (hence on any interval of length 2π, because of periodicity). By definition, this means that the integral of the product of any two different of these functions over that interval is zero; in formulas, for any integers m and $n \neq m$ we have

$$\int_{-\pi}^{\pi} \cos mx \; \cos nx \; dx = 0 \, (m \neq n) \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx \; \sin nx \; dx = 0 \, (m \neq n) \, ,$$

and for any integers m and n (including $m = n$) we have

$$\int_{-\pi}^{\pi} \cos mx \; \sin nx \; dx = 0.$$

This is the most important property of the trigonometric system, the key in driving the Euler formulas (where we proved this orthogonality).

Convergence and Sum of Fourier Series

We now present a theorem on the convergence and the sum of Fourier series : Suppose that $f(x)$ is any given periodic function of period 2π for which the integrals in Euler formulas exist; for instance, $f(x)$ is continuous or merely piecewise continuous (continuous except for finitely many finite jumps in the interval of integration). Then we can compute the Fourier coefficients of $f(x)$ and use them to form the Fourier series of $f(x)$.

It would be nice if the series thus obtained converged and had the sum $f(x)$. Most functions appearing in applications are such that this is true (except at jumps of $f(x)$, which we discus below). In this case, in which the Fourier series of $f(x)$ does represent $f(x)$, we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with an equality sign.

Convergence and Sum of Fourier Series

If the Fourier series of $f(x)$ does not have the sum $f(x)$ or does not converge, one still writes

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with a tilde $\sim$, which indicates that the trigonometric series on the right has the Fourier coefficients of $f(x)$ as its coefficients, so it is the Fourier series of $f(x)$.

The class of functions that can be represented by Fourier series is surprisingly large and general.

Left-and right-hand derivatives

The **left-hand limit** of $f(x)$ at x_0 is defined as the limit of $f(x)$ as x approaches x_0 from the left and is frequently denoted by $f(x_0 - 0)$. Thus

$$f(x_0 - 0) = \lim_{h \to 0} f(x_0 - h)$$

as $h \to 0$ through positive values. The **right-hand limit** is denoted by $f(x_0 + 0)$ and

$$f(x_0 - 0) = \lim_{h \to 0} f(x_0 + h)$$

as $h \to 0$ through positive values. The **left-and right-hand derivatives** of $f(x)$ at x_0 are defined as the limit of

$$\frac{f(x_0 - h) - f(x_0 - 0)}{-h} \text{and} \frac{f(x_0 + h) - f(x_0 + 0)}{h}$$

respectively, as $h \to 0$ through positive values. Of course if $f(x)$ is continuous at x_0 the last term in both numerators is simply $f(x_0)$.

Representation by a Fourier series

Theorem 2.

If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \leq x \leq \pi$ and has a left-hand derivative and right-hand derivative at each point of that interval, then the Fourier series of $f(x)$ [with coefficients in Euler formulas] is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left-and right-hand limits of $f(x)$ at x_0.

Proof of convergence theorem

We prove convergence for a continuous function $f(x)$ having continuous first and second derivatives. Integrating

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx n = 1, 2, \cdots$$

by parts, we get

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\;dx = \frac{f(x)\sin nx}{n\pi}\bigg|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\;dx.$$

The first term on the right is zero. Another integration by parts gives

$$a_n = \frac{f'(x)\cos nx}{n^2\pi}\bigg|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx\;dx.$$

Proof of convergence theorem (contd...)

The first term on the right is zero because of the periodicity and continuity of $f'(x)$. Since f'' is continuous in the interval of integration, we have

$$|f''(x)| < M$$

for an appropriate constant M.

Furthermore, $|\cos nx| \leq 1$. It follows that

$$|a_n| = \frac{1}{n^2\pi} \left| \int_{-\pi}^{\pi} f''(x) \cos nx \; dx \right| < \frac{1}{n^2\pi} \int_{-\pi}^{\pi} M \; dx = \frac{2M}{n^2}.$$

Proof of convergence theorem (contd...)

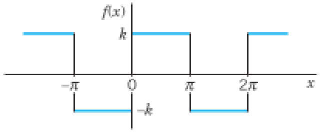
Similarly, $|b_n| < 2\ M/n^2$ for all n. Hence the absolute value of each term of the Fourier series of $f(x)$ is at most equal to the corresponding term of the series

$$|a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$$

which is convergent. Hence that Fourier series converges and the proof is complete.

Note : The proof of convergence in the case of a piecewise continuous function $f(x)$ and the proof that under the assumptions in the theorem the Fourier series represents $f(x)$ are substantially more complicated.

Example 3.



Given function $f(x)$ (Periodic reactangular wave)

The square wave has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is k. Hence the average of these limits is 0. The Fourier series

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right).$$

of the square wave does indeed converge to this value when $x = 0$ because then all its terms are 0. Similarly for the other jumps.

Summary

A Fourier series of a given function $f(x)$ of period 2π is a series of the form

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with coefficients given by the Euler formulas.

Theorem (2) gives conditions that are sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

Exercise 4.

Showing the details of your work, find the Fourier series of the function $f(x)$, which is assumed to have the period 2π, and plot accurate graphs of the firs three partial sums, where $f(x)$ equals

1. $f(x) = x^3\,(\pi < x < \pi)$
2. $f(x) = x + |x|\,(-\pi < x < \pi)$
3. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ -1 & \text{if } 0 < x < \pi \end{cases}$
4. $f(x) = \begin{cases} -1 & \text{if } 0 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 2\pi \end{cases}$
5. $f(x) = \begin{cases} 1 & \text{if } -\pi/2 < x < \pi/2 \\ -1 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

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Showing the details of your work, find the Fourier series of the function $f(x)$, which is assumed to have the period 2π, and plot accurate graphs of the firs three partial sums, where $f(x)$ equals

1. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
2. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
3. $f(x) = \begin{cases} x^2 & \text{if } -\pi/2 < x < \pi/2 \\ \pi^2/4 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

Functions of Any Period $p = 2L$

The functions considered so far had period 2π, for simplicity. Of course, in applications, periodic functions will generally have other periods. But we show that the transition from period $p = 2\pi$ to period $p = 2L$ is quite simple. It amounts to a stretch (or contraction) of scale on the axis. If a function $f(x)$ of period $p = 2L$ has a **Fourier series**, we claim that this series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \tag{12}$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

$$\begin{aligned} (a) \quad & a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\ (b) \quad & a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx n = 1, 2, \cdots \\ (c) \quad & b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx n = 1, 2, \cdots . \end{aligned} \tag{13}$$

The series with *arbitrary* coefficient is called a **trigonometric series**, and Theorem 2 extends to any period p.

Proof. By setting $v = \pi x/L$, we get $x = Lv/\pi$. Also, $x = \pm L$ corresponds to $v = \pm\pi$. Thus f, regarded as a function of v that we call $g(v)$,

$$f(x) = g(v),$$

has period 2π. Accordingly, this 2π-periodic function $g(v)$ has the Fourier series

$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv) \tag{14}$$

with coefficients

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)\,dv \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\,dv \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin nv\,dv. \end{aligned} \tag{15}$$

Functions of Any Period $p = 2L$

Since $v = \pi x/L$ and $g(v) = f(x)$, formula (14) gives (1). In (15) we introduce $x = Lv/\pi$ as variable of integration.

Then the limits of integration $v = \pm\pi$ become $x = \pm L$. Also, $v = \pi x/L$ implies $dv = \pi dx/L$. Thus $dv/2\pi = dx/2L$ in a_0. Similarly, $dv/\pi = dx/L$ in a_n and b_n.

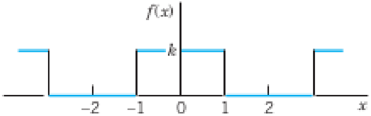
Hence (15) gives (13).

Interval of integration. In (13) we may replace the interval of integration by any interval of length $p = 2L$, for example, by the interval $0 \leq x \leq 2L$.

Example 6.

Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases} \qquad p = 2L = 4, L = 2.$$



Solution.

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{k}{2}.$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if } n = 1, 5, 9, \ldots, a_n = -2k/n\pi \quad \text{if } n = 3, 7, 11, \ldots.$$

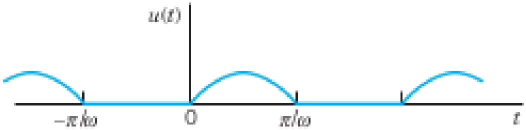
$b_n = 0$ for $n = 1, 2, \cdots$. Hence the result is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - \cdots\right).$$

Example 7.

A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0, \\ E \sin \omega t & \text{if } 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, L = \frac{\pi}{\omega}.$$



Solution. Since $u = 0$ when $-L < t < 0$, with t instead of x,

$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\sin\omega t\, dt = \frac{E}{\pi}$$

and with $x = \omega t$ and $y = n\omega t$,

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t\cos n\omega t\, dt = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[\sin(1+n)\omega t + \sin(1-n)\omega t]\, dt.$$

If $n = 1$, the integral on the right is zero, and if $n = 2, 3, \ldots$, we readily obtain

$$\begin{aligned} a_n &= \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega} \\ &= \frac{E}{2\pi}\left(\frac{-\cos(1+n)\pi + 1}{(1+n)} + \frac{-\cos(1-n)\pi + 1}{(1-n)}\right). \end{aligned}$$

In n is odd, this is equal to zero, and for even n we have

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi} \qquad (n = 2, 4, \cdots).$$

In a similar fashion we find that $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \ldots$, Consequently.

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right).$$

Exercise 8.

Find the Fourier series of the periodic function $f(x)$, of period $p = 2L$, and sketch $f(x)$ and the first three partial sums. (Show that details of your work.)

1. $f(x) = -1\ (-1 < x < 0)$, $f(x) = 1\ (0 < x < 1)$, $p = 2L = 2$
2. $f(x) = 1\ (-1 < x < 0)$, $f(x) = -1\ (0 < x < 1)$, $p = 2L = 2$
3. $f(x) = 0\ (-2 < x < 0)$, $f(x) = 2\ (0 < x < 2)$, $p = 2L = 4$
4. $f(x) = |x|\ (-2 < x < 2)$, $p = 2L = 4$
5. $f(x) = 2x\ (-1 < x < 1)$, $p = 2L = 2$
6. $f(x) = 1 - x^2\ (-1 < x < 1)$, $p = 2L = 2$
7. $f(x) = 3x^2\ (-1 < x < 1)$, $p = 2L = 2$
8. $f(x) = \frac{1}{2} + x\ (-\frac{1}{2} < x < 0)$, $f(x) = \frac{1}{2} - x\ (0 < x < \frac{1}{2})$, $p = 2L = 1$

Exercise 9.

Find the Fourier series of the periodic function $f(x)$, of period $p = 2L$, and sketch $f(x)$ and the first three partial sums. (Show that details of your work.)

1. $f(x) = 0$, $(-1 < x < 0)$, $f(x) = x$ $(0 < x < 1)$, $p = 2L = 2$
2. $f(x) = x$ $(0 < x < 1)$, $f(x) = 1 - x$ $(1 < x < 2)$, $p = 2L = 2$
3. $f(x) = \pi \sin \pi x$ $(0 < x < 1)$, $p = 2L = 1$
4. $f(x) = \pi x^3/2$ $(-1 < x < 1)$, $p = 2L = 2$
5. (Rectifier) *Find the Fourier series of the periodic function that is obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier.*

The periodic square wave function was even and had only cosine terms in its Fourier series, no sine terms. This is typical. In fact, unnecessary work (and corresponding sources of errors) in determining Fourier coefficients can be avoided if a function is even or odd.

A function $y = g(x)$ is **even** if

$$g(-x) = g(x) \text{for all} \, x.$$

The graph of such a function is symmetric with respect to the y-axis.

A function $h(x)$ is **odd** if

$$h(-x) = -h(x) \text{for all} \, x.$$

The function $\cos nx$ is even, while $\sin nx$ is odd.

Three Key Facts

1. If $g(x)$ is an **even** function, then
$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx \qquad (g \text{ even}).$$
2. If $h(x)$ is an **odd** function, then
$$\int_{-L}^{L} h(x)\,dx = 0 \qquad (h \text{ odd}).$$
3. The product of an even and an odd function is odd.

Fourier cosine series, Fourier sine series

Theorem 10.

The Fourier series of an even *function of period* $2L$ *is a* Fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

with coefficients

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots .$$

Theorem 11.

The Fourier series of an odd *function of period* $2L$ *is a* Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

with coefficients

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

The case of period 2π

The above theorems give for an even function simply

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad (f \text{ even})$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \qquad n = 1, 2, \ldots.$$

Similarly, for an odd 2π-periodic function we simply have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (f \text{ odd})$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, \ldots.$$

Sum of Functions

Theorem 12.

The Fourier coefficients of a sum $f_1 + f_2$ *are the sums of the corresponding Fourier coefficients of* f_1 *and* f_2*. The Fourier coefficients of* cf *are* c *times the corresponding Fourier coefficients of* f.

Rectangular pulse

The function $f^*(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ 2k & \text{if } 0 < x < \pi \end{cases}$ is the sum of the function

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$$

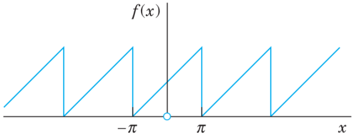
and the constant k. Hence, from Theorem (12) we conclude that

$$f^*(x) = k + \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

Example 13.

Find the Fourier series of the function

$$f(x) = x + \pi \quad \text{if} -\pi < x < \pi \quad \text{and} \quad f(x + 2\pi) = f(x).$$



Solution. We may write

$$f = f_1 + f_2, \text{ where } f_1 = x \text{ and } f_2 = \pi.$$

The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π. Hence, by Theorem 12, the Fourier coefficients a_n, b_n are those of f_1, except for a_0, which is π.

Sawtooth wave

Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \ldots$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx.$$

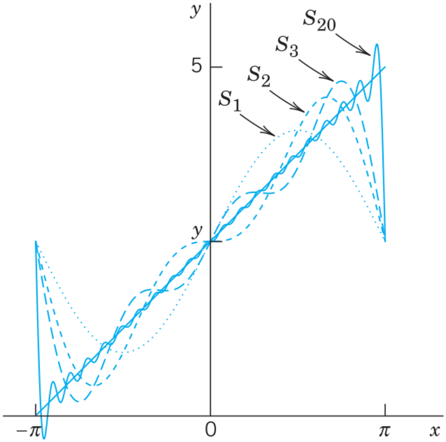
Integrating by parts we obtain

$$b_n = \frac{2}{\pi}\left[\frac{-x\cos nx}{n}\bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right] = \frac{12}{n}\cos n\pi.$$

Hence $b_1 = 2$, $b_2 = -2/2$, $b_3 = 2/3$, $b_4 = -2/4$, $\ldots$, and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right).$$

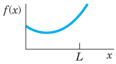
Sawtooth wave - Partial sums



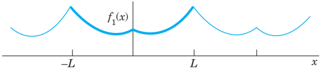
Partial sums S_1, S_2, S_3, S_{20}

Half-Range Expansions

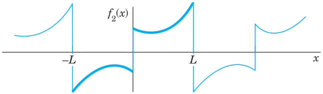
Half-range expansions are Fourier series. This concerns a practically useful simple idea. In applications we often want to employ a Fourier series for a function f that is given only on some interval, say, $0 \leq x \leq L$.



(0) The given function $f(x)$



(a) $f(x)$ continued as an ***even*** periodic function of period $2L$



(b) $f(x)$ continued as an ***odd*** periodic function of period $2L$

Fig. 270. Even and odd extensions of period $2L$

Half-Range Expansions

This function f can be the displacement of a violin string of (undistorted) length L or the temperature in a metal bar of length L, and so on. Now the key idea is as follows.

For our function f we can calculate Fourier sine or cosine series. And we have a choice. If we use Theorem 10, we get a Fourier cosine series. This series represents the **even periodic extension** f_1 of f in Fig. 270(a).

If in a practical problem we think that using Theorem 11 is better, we get a Fourier sine series. This series represents the **odd periodic extension** f_2 of f in Fig. 270(b). Both extensions have period $2L$.

This motivates the name **half-range expansions:** f is given (and of physical interest) only on half the range, half the interval of periodically of length $2L$.

Triangle and its half-range expansions

Example 14.

Find the two half-range expansions of the function

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L. \end{cases}$$

Solution. (a) **Even periodic extension.** From (4) we obtain

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x dx + \frac{2k}{L}\int_{L/2}^{L} (L-x)\, dx\right] = \frac{k}{2},$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x \cos\frac{n\pi}{L}x\, dx + \frac{2k}{L}\int_{L/2}^{L} (L-x)\cos\frac{n\pi}{L}x\, dx\right].$$

Triangle and its half-range expansions

We consider a_n. For the first integral we obtain by integration by parts

$$\int_0^{L/2} x \cos \frac{n\pi}{L} x \, dx = \frac{Lx}{n\pi} \sin \frac{n\pi}{L} x \Big|_0^{L/2} - \frac{L}{n\pi} \int_0^{L/2} \sin \frac{n\pi}{L} x \, dx$$
$$= \frac{L^2}{2n\pi} \sin \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \left(\cos \frac{n\pi}{2} - 1 \right).$$

Similarly, for the second integral we obtain

$$\int_{L/2}^{L} (L - x) \cos \frac{n\pi}{L} x \, dx = \frac{L}{n\pi} (L - x) \sin \frac{n\pi}{L} x \Big|_{L/2}^{L} + \frac{L}{n\pi} \int_{L/2}^{L} \sin \frac{n\pi}{L} x \, dx$$
$$= -\frac{L}{n\pi} \left(L - \frac{L}{2} \right) \sin \frac{n\pi}{2} - \frac{L^2}{n^2\pi^2} \left(\cos n\pi - \cos \frac{n\pi}{2} \right).$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus,

$$a_2 = -16k/2^2\pi^2, \qquad a_6 = -16k/6^2\pi^2, \qquad a_{10} = -16k/10^2\pi^2, \ldots,$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \ldots$. Hence the first half-range expansion of $f(x)$ is

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extensions of the given function $f(x)$, of period $2L$, shown in Fig 270(a).

(b) Odd periodic extension. Similarly, we obtain

$$b_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}. \tag{16}$$

Hence the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8k}{\pi^2} \left(\frac{1}{1^2} \sin \frac{\pi}{L}x - \frac{1}{3^2} \sin \frac{3\pi}{L}x + \frac{1}{5^2} \sin \frac{5\pi}{L}x - + \cdots \right).$$

This series represents the odd periodic extension of $f(x)$, of period $2L$, shown in Fig. 270(b).

Exercises 15.

1. Even and Odd Functions : *Are the following functions odd, even, or neither odd nor even?*
   (a) $|x^3|$, $x\cos nx$, $x^2\cos nx$, $\cos hx$, $\sin hx$, $\sin x + \cos x$, $x|x|$
   (b) $x + x^2$, $|x|$, e^x, e^{x^2}, $\sin^2 x$, $x\sin x$, $\ln x$, $x\cos x$, $e^-|x|$
2. *Are the following functions* $f(x)$*, which are assumed to be periodic, of period* 2π*, even, odd or neither even nor odd?*
   (a) $f(x) = x^2\ (0 < x < 2\pi)$
   (b) $f(x) = x^4\ (0 < x < 2\pi)$
   (c) $f(x) = e^{-|x|}\ (-\pi < x < \pi)$
   (d) $f(x) = |sin 5x|\ (-\pi < x < \pi)$
   (e) $f(x) = \begin{cases} 0 & if\, 2 < x < 2\pi - 2 \\ x & if\, -2 < x < 2 \end{cases}$
   (f) $f(x) = \begin{cases} \cos^2 x & if\, -\pi < x < 0 \\ \sin^2 x & if\, 0 < x < \pi \end{cases}$
   (g) $f(x) = x^3\ (-\pi/2 < x < 3\pi/2)$

Exercises 16.

1. *Are the following expressions even or odd? Sums and products of even functions and of odd functions. Products of even times odd functions. Absolute values of odd functions.* $f(x) + f(-x)$ *and* $f(x) - f(-x)$ *for arbitrary* $f(x)$.
2. *Write* e^{kx}, $1/(1-x)$, $\sin(x+k)$, $\cosh(x+k)$ *as sums of an even and an odd function.*
3. *Find all functions that are both even and odd.*
4. *Is* $\cos^3 x$ *even or odd?* $\sin^3 x$*? Find the Fourier series of these two functions. Do you recognize familiar identities?*

Exercises 17.

State whether the given function is even or odd. Find its Fourier series. Sketch the function and some partial sums.

1. $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
2. $f(x) = \begin{cases} -2x & \text{if } -\pi < x < 0 \\ 2x & \text{if } 0 < x < \pi \end{cases}$
3. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2. \end{cases}$
4. $f(x) = \begin{cases} x & \text{if } 0 < x < \pi \\ \pi - x & \text{if } \pi < x < 2\pi \end{cases}$
5. $f(x) = x^2/2 \ (-\pi < x < \pi)$
6. $f(x) = 3x(\pi^2 - x^2) \ (-\pi < x < \pi)$

Exercises 18.

Show that

1. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.
2. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$.
3. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \frac{\pi^2}{12}$.

Exercises 19.

Find the Fourier cosine series as well as the Fourier sine series. Sketch $f(x)$ and its two periodic extensions.

1. $f(x) = 1 \ (0 < x < L)$
2. $f(x) = x \ (0 < x < L)$
3. $f(x) = x^2 \ (0 < x < L)$
4. $f(x) = \pi - x \ (0 < x < \pi)$
5. $f(x) = x^3 \ (0 < x < L)$
6. $f(x) = e^x \ (0 < x < L)$

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